FREE ELASTIC WAVES ON THE SURFACE OF A TUBE OF INFINITE THICKNESS

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PMM Vol.27, No.3, 1963, pp.551-554

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(Received January 29, 1962)

We are investigating the three-dimensional case of free oscillations propagated on the surface of an infinite circular cylinder which represents a cavity in infinite elastic space. Suppose the infinite elastic space has a cavity of the shape of an infinitely long circular cylinder with the diameter $2R$, whose axis we assume to be the z-axis, and r, θ are the polar coordinates of the points in a plane perpendicular to the axis of the cylinder. The vector equation of motion of the homogeneous and isotropic elastic medium, when mass forces are absent, has the form

$$(\lambda + 2\mu)\,\mathrm{grad\,div}\,\mathbf{u} - \mu\,\mathrm{rot\,rot}\,\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{1}$$

where $\mathbf{u}$ is the displacement vector, λ, μ are Lamé's constants, ρ is the density. Let us decompose the displacement vector $\mathbf{u}$ into a sum of two vectors – the potential and the solenoidal

$$\mathbf{u} = \mathrm{grad}\,\Phi + \mathrm{rot}\,\boldsymbol{\psi} \tag{2}$$

Then equation (1) will be satisfied if we set

$$\nabla^2\Phi = \frac{1}{a^2}\frac{\partial^2\Phi}{\partial t^2}, \qquad \nabla^2\boldsymbol{\psi} = \frac{1}{b^2}\frac{\partial^2\boldsymbol{\psi}}{\partial t^2} \qquad \left(a = \sqrt{\frac{\lambda+2\mu}{\rho}},\ b = \sqrt{\frac{\mu}{\rho}}\right) \tag{3}$$

where ∇ is the Hamilton's operator. Formula (2) represents the general solution of equation (1). The vector potential can always be chosen so as to have its z component equal to zero. Expressing the gradient and curl in cylindrical coordinates and bearing in mind that $\nabla^2\boldsymbol{\psi} = \mathrm{grad\,div}\,\boldsymbol{\psi} - \mathrm{rot\,rot}\,\boldsymbol{\psi}$, we obtain

$$u_r=\frac{\partial\Phi}{\partial r}-\frac{\partial\psi_\varphi}{\partial z},\quad u_\varphi=\frac{1}{r}\frac{\partial\Phi}{\partial\varphi}+\frac{\partial\psi_r}{\partial z},\quad u_z=\frac{\partial\Phi}{\partial z}+\frac{1}{r}\frac{\partial(r\psi_\varphi)}{\partial r}-\frac{1}{r}\frac{\partial\psi_r}{\partial\varphi} \tag{4}$$

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{\partial^2\Phi}{\partial z^2}=\frac{1}{a^2}\frac{\partial^2\Phi}{\partial t^2}$$

$$\frac{\partial^2\psi_r}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_r}{\partial r}+\frac{\partial^2\psi_r}{\partial z^2}-\frac{1}{r^2}\left(\psi_r-\frac{\partial^2\psi_r}{\partial\varphi^2}\right)-\frac{2}{r^2}\frac{\partial\psi_\varphi}{\partial\varphi}=\frac{1}{b^2}\frac{\partial^2\psi_r}{\partial t^2} \tag{5}$$

$$\frac{\partial^2\psi_\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_\varphi}{\partial r}+\frac{\partial^2\psi_\varphi}{\partial z^2}-\frac{1}{r^2}\left(\psi_\varphi-\frac{\partial^2\psi_\varphi}{\partial\varphi^2}\right)+\frac{2}{r^2}\frac{\partial\psi_r}{\partial\varphi}=\frac{1}{b^2}\frac{\partial^2\psi_\varphi}{\partial t^2}$$

For the components of stress acting on an element of the boundary area we have

$$\sigma_r=\lambda\operatorname{div}\mathbf{u}+2\mu\frac{\partial u_r}{\partial r},\quad \tau_{r\varphi}=\mu\left\{\frac{1}{r}\frac{\partial u_r}{\partial\varphi}+r\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right\},\quad \tau_{rz}=\mu\left\{\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right\} \tag{6}$$

where u_r, u_φ, u_z are defined by the expressions (4). Let us set, as does Terezawa [1]

$$\psi_r+i\psi_\varphi=\psi_1,\qquad \psi_r-i\psi_\varphi=\psi_2\qquad (i^2=-1) \tag{7}$$

Then

$$\frac{\partial^2\psi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_1}{\partial r}+\frac{\partial^2\psi_1}{\partial z^2}-\frac{1}{r^2}\left(\psi_1-\frac{\partial^2\psi_1}{\partial\varphi^2}\right)+\frac{2i}{r^2}\frac{\partial\psi_1}{\partial\varphi}=\frac{1}{b^2}\frac{\partial^2\psi_1}{\partial t^2}$$

$$\frac{\partial^2\psi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_2}{\partial r}+\frac{\partial^2\psi_2}{\partial z^2}-\frac{1}{r^2}\left(\psi_2-\frac{\partial^2\psi_2}{\partial\varphi^2}\right)-\frac{2i}{r^2}\frac{\partial\psi_2}{\partial\varphi}=\frac{1}{b^2}\frac{\partial^2\psi_2}{\partial t^2} \tag{8}$$

Thus, from the system of equations (5) by means of the substitution (7) we have obtained equations (8), independent of each other. Let

$$\Phi=\Phi_1e^{ipt}e^{in\varphi} \tag{9}$$

Then

$$\frac{\partial^2\Phi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi_1}{\partial r}+\left(h^2-\frac{n^2}{r^2}\right)\Phi_1+\frac{\partial^2\Phi_1}{\partial z^2}=0\qquad\left(h^2=\frac{p^2}{a^2}\right). \tag{10}$$

Substituting $\Phi_1=\Phi_2(r)e^{-i\theta z}$ in (10), we have

$$r^2\frac{d^2\Phi_2}{dr^2}+r\frac{d\Phi_2}{dr}-[(\theta^2-h^2)r^2+n^2]\Phi_2=0 \tag{11}$$

Considering the oscillations which are rapidly damped in depth, let us introduce into the integral of equation (11) the Macdonald's function $K_n[\sqrt{(\theta^2-h^2r)}]$, which, as is well known, for $\sqrt{(\theta^2-h^2)}>0$ and $r\to\infty$ tends to zero exponentially. Thus, for $r\geqslant R$ we have the solution of the first of equations (5) in the form

$$\Phi=AK_n(\sqrt{\theta^2-h^2}r)e^{in\varphi}e^{i(pt-\theta z)} \tag{12}$$

If one considers oscillations characterized by beats then the function of Weber-Neumann should be taken as the integral of equation (11). Analogously to the solution (12) found above, we will obtain the solution of equations (8) in the form ($k^2 = p^2/b^2$)

$$\psi_1 = BK_{n+1}(\sqrt{\theta^2 - k^2}r)\, e^{in\varphi} e^{i(pt-\theta z)}, \qquad \psi_2 = CK_{n-1}(\sqrt{\theta^2 - k^2}r)\, e^{in\varphi} e^{i(pt-\theta z)} \tag{13}$$

In formulas (12) and (13) the quantities A, B, C are constants. From formula (13) we obtain the expressions for the components ψ_r, ψ_φ of the vector potential

$$\begin{aligned} \psi_r &= \frac{\psi_1 + \psi_2}{2} = \frac{e^{i(pt-\theta z)}}{2}\{BK_{n+1}(\beta r) + CK_{n-1}(\beta r)\}\, e^{in\varphi} \\ \psi_\varphi &= \frac{\psi_1 - \psi_2}{2i} = \frac{e^{i(pt-\theta z)}}{2i}\{BK_{n+1}(\beta r) - CK_{n-1}(\beta r)\}\, e^{in\varphi} \end{aligned} \tag{14}$$

We write the scalar potential in the form

$$\Phi = Ae^{i(pt-\theta z)} K_n(\alpha r)\, e^{in\varphi} \qquad (\alpha = \sqrt{\theta^2 - h^2}, \quad \beta = \sqrt{\theta^2 - k^2}) \tag{15}$$

Substituting formulas (14) and (15) into (4), and the expressions thus obtained into (6), and making use of the recurrence formulas for the functions $K_n(x)$

$$\begin{gathered} K_n'(x) = -{}^1/_2\,[K_{n+1}(x) + K_{n-1}(x)], \qquad xK_n'(x) = -nK_n(x) - xK_{n+1}(x) \\ 2K_n''(x) = {}^1/_2K_{n-2}(x) + K_n(x) + {}^1/_2K_{n+2}(x) \end{gathered} \tag{16}$$

we obtain the expressions for the projections of the displacement vector and the components of the stress vector on the surface of the cavity with $r = R$ (17)

$$u_r^\circ = -{}^1/_2A\alpha\,\{[K_{n+1}(\alpha R) + K_{n-1}(\alpha R)]\, e^{i(pt-\theta z)} e^{in\varphi} + {}^1/_2\theta\,[BK_{n+1}(\beta R) - CK_{n-1}(\beta R)]\}$$

$$\begin{aligned} u_\varphi^{(0)} &= Ain/RK_n(\alpha R)\, e^{i(pt-\theta z)} e^{in\varphi} - {}^1/_2 i\theta\,[BK_{n+1}(\beta R) + CK_{n-1}(\beta R)]\, e^{i(pt-\theta z)} e^{in\varphi} \\ u_z^{(0)} &= -Ai\theta K_n(\alpha R)\, e^{i(pt-\theta z)} e^{in\varphi} + {}^1/_2\beta i\,(B - C)\, K_n(\beta R)\, e^{i(pt-\theta z)} e^{in\varphi} \\ [\tau_{rz}]\,|_{r=R} &= \mu\,\{Ai\theta\alpha\,[K_{n+1}(\alpha R) + K_{n-1}(\alpha R)]\, e^{i(pt-\theta z)} e^{in\varphi} - \\ &\quad - {}^1/_4 iB\,[(2\theta^2 + \beta^2)\, K_{n+1}(\beta R) + \beta^2 K_{n-1}(\beta R)]\, e^{i(pt-\theta z)} e^{in\varphi} + \\ &\quad + {}^1/_4 iC\,[(2\theta^2 + \beta^2)\, K_{n-1}(\beta R) + \beta^2 K_{n+1}(\beta R)]\, e^{i(pt-\theta z)} e^{in\varphi} \\ [\tau_{r\varphi}]_{r=R} &= \mu\,\{(-2ni/R)\, A\,[(n+1)\, K_n(\alpha R)/R + \alpha K_{n-1}(\alpha R)] + \\ &\quad + {}^1/_2\theta i\beta BK_{n+2}(\beta R) + {}^1/_2\theta i\beta CK_{n-2}(\beta R)\}\, e^{i(pt-\theta z)} e^{in\varphi} \\ [\sigma_r]_{r=R} &= \mu\,\{A\,[(2h^2 - k^2 + \alpha^2)\, K_n(\alpha R) + {}^1/_2\alpha^2 K_{n-2}(\alpha R) + {}^1/_2\alpha^2 K_{n+2}(\alpha R)] + \\ &\quad + \theta\beta\,\{-{}^1/_2B\,[K_{n+2}(\beta R) + K_n(\beta K)]\} + {}^1/_2C\,[K_n(\beta R) + K_{n-2}(\beta R)]\}\, e^{i(pt-\theta z)} e^{in\varphi} \end{aligned} \tag{18}$$

To obtain the condition of stress-free surface of the cylinder we set in (19) σ_r, τ_{rz}, $\tau_{r\varphi}$ equal to zero for $r = R$. We then obtain three linear homogeneous equations for determination of the constants A, B, C.

For the existence of solutions A, B, C, different from zero, it is necessary that the third-order determinant $\Delta(\theta) = |a_{ij}|$ be equal to zero. The terms of the determinant a_{ij} are of the following form

$$a_{11} = \theta\alpha\,[K_{n+1}(\alpha R) + K_{n-1}(\alpha R)], \qquad a_{12} = -\frac{1}{4}\,[(2\theta^2 + \beta^2)\,K_{n+1}(\beta R) + \beta^2 K_{n-1}(\beta R)$$

$$a_{13} = \frac{1}{4}\,[(2\theta^2 + \beta^2)\,K_{n-1}(\beta R) + \beta^2 K_{n+1}(\beta K)], \qquad a_{22} = \frac{1}{2}\,\theta\beta K_{n+2}(\beta R)$$

$$a_{21} = -\,(2n\,/\,R^2)\,[(n+1)\,K_n(\alpha R) + \alpha R K_{n-1}(\alpha R], \qquad a_{23} = \frac{1}{2}\,\theta\beta K_{n-2}(\beta R)$$

$$a_{31} = (2h^2 - k^2 + \alpha^2)\,K_n(\alpha R) + \frac{1}{2}\,\alpha^2 K_{n-2}(\alpha R) + \frac{1}{2}\,\alpha^2 K_{n+2}(\alpha R)$$

$$a_{32} = -\,\frac{1}{2}\,\theta\beta\,[K_{n+2}(\beta R) + K_n(\beta R], \qquad a_{33} = \frac{1}{2}\,\theta\beta\,[K_n(\beta R) + K_{n-2}(\beta R)]$$

Thus the equation of frequencies has the form

$$\Delta(\theta) = |\,a_{ij}\,| = 0 \tag{20}$$

The third-order determinant yields a transcendental equation for the determination of θ. Let us consider oscillations with high frequencies. We designate by $\vartheta = \theta/p$ the quantity reciprocal to the velocity of wave propagation along the cylinder.

Assuming that the frequency is sufficiently high, we replace in equation (20) the functions $K_n(x)$ by their asymptotic expressions. We have

$$K_n(x) \sim \sqrt{\frac{\pi}{2}}\,\frac{1}{\sqrt{x}}\,e^{-x} \tag{21}$$

Substituting (21) into (20) we find

$$\Delta(\theta) \sim \left(\sqrt{\frac{\pi}{2}}\right)^3 \begin{vmatrix} \dfrac{2\theta\alpha e^{-\alpha R}}{\sqrt{\alpha R}} & \dfrac{-\,{}^1/_2\,(\theta^2 + \beta^2)\,e^{-\beta R}}{\sqrt{\beta R}} & \dfrac{{}^1/_2\,(\theta^2 + \beta^2)\,e^{-\beta R}}{\sqrt{\beta R}} \\ \dfrac{-\,2ne^{-\alpha R}}{R^2\sqrt{\alpha R}}\,[(n+1) + \alpha R] & \dfrac{\theta\beta e^{-\beta R}}{2\sqrt{\beta R}} & \dfrac{\theta\beta e^{-\beta R}}{2\sqrt{\beta R}} \\ \dfrac{(2h^2 - k^2 + 2\alpha^2)\,e^{-\alpha R}}{\sqrt{\alpha R}} & \dfrac{-\,\theta\beta e^{-\beta R}}{\sqrt{\beta R}} & \dfrac{\theta\beta e^{-\beta R}}{\sqrt{\beta R}} \end{vmatrix} \tag{22}$$

Adding the second and third columns in (22) we easily find

$$\Delta(\theta) \sim \left(\sqrt{\frac{\pi}{2}}\right)^3 \frac{1}{\sqrt{\alpha R}}\,\frac{1}{\sqrt{\beta R}}\,e^{-\alpha R}e^{-\beta R} \left\{\frac{-\,\theta\sqrt{\theta^2 - k^2}}{2}\;p^4\left[\left(2\,\frac{\theta^2}{p^2} - \frac{1}{b^2}\right)^2 - \right.\right.$$

$$\left.\left. -\,4\,\frac{\theta^2}{p^2}\sqrt{\frac{\theta^2}{p^2} - \frac{1}{a^2}}\sqrt{\frac{\theta^2}{p^2} - \frac{1}{b^2}}\right]\right\} \tag{23}$$

The expression in square brackets (23) is called the Rayleigh function [2]. As is well known, the equation

$$(2\vartheta^2 - b^{-2})^2 - 4\vartheta^2 \sqrt{\vartheta^2 - a^{-2}} \sqrt{\vartheta^2 - b^{-2}} = 0 \qquad (24)$$

has a single real positive root and another one negative of equal magnitude. Thus, when the frequency tends to infinity, equation (20) has a single real positive root and another one negative of equal modulus. These roots are the roots of equation (24), known as the Rayleigh's equation. Let us designate these roots by $\vartheta = \pm 1/c$, where c is the velocity of the Rayleigh wave.

Thus we arrive at the result: when the frequency tends to infinity the velocity of wave propagation on the surface of a cylinder tends to the velocity of the Rayleigh wave.

For the case of oscillations with axial symmetry, equation $\Delta(\theta) = 0$ has been investigated in [3].

Let us note that the method considered here can also be applied in the investigation of free oscillations of a cylinder. For that purpose, in the expressions (12) and (14) the modified Bessel functions [4] $I_{n+1}(\xi)$, $I_n(\xi)$, $I_{n-1}(\xi)$ should be taken instead of the Macdonald functions $K_{n-1}(\xi)$, $K_n(\xi)$, $K_{n-1}(\xi)$.

In that case the damping along the depth of the cylinder will also take place, due to the properties of the function $I_n(\xi)$. The condition of stress-free surface of the cylinder will lead to an equation analogous to $\Delta(\theta) = 0$, which has been investigated for the case of oscillations with axial symmetry [5].

BIBLIOGRAPHY

1. Terezawa, Oscillations of the deep-sea surface caused by a local disturbance. *Sci. Rept. Tōhoku Imp. Univ.*, Vol. 13, S, 1917.
2. Rayleigh, Waves propagated along plane surface of an elastic solid. *Proc. Math. Soc.*, Vol. 17, 1886.
3. Mindlin, Ia.A., Rasprostranenie voln po poverkhnosti beskonechno dlinnogo krugovogo tsilindra, predstavliaiushchego soboi vyrez v beskonechnom uprugom prostranstve (Propagation of waves on the surface of an infinitely long circular cylinder which represents a cavity in the infinite elastic space). *Dokl. Akad. Nauk SSSR*, Vol. 42, No. 4, 1944.

4. Sneddon, I.N. and Berry, D.S., *Klassicheskaia teoriia uprugosti (The Classical Theory of Elasticity)*. IL, 1961.

5. Mindlin, Ia.A., Rasprostranenie voln po poverkhnosti beskonechno dlinnogo krugovogo tsilindra (Propagation of waves on the surface of an infinitely long circular cylinder). *Dokl. Akad. Nauk SSSR*, Vol. 52, No. 2, 1946.

Translated by O.S.